Approximations and Perturbations in Nonlocal Reaction-Diffusion Equations

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Abstract: Singular perturbation problems for nonlocal reaction-diffusion equations with the Neumann boundary conditions are treated. It is exhibited, by means of approximation and perturbation methods, that the layer solutions of a scalar bistable nonlocal reaction-diffusion equation converge to solutions of the averaged mean curvature flow on a finite time interval as the singular perturbation parameter tends to zero.

Keywords: Approximation, perturbation, nonlocal reaction-diffusion equation, internal transition layer, interface, the averaged mean curvature flow.

1. Nonlocal reaction-diffusion equation

We consider in this paper the following scalar bistable nonlocal reaction-diffusion equation:

$$\text{(RD)} \qquad \begin{cases} u_t = \epsilon^2 \triangle u + f(u) - \langle f(u) \rangle, \\ \partial u / \partial \mathbf{n} = 0. \end{cases}$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with total volume $|\Omega|$ and the outward unit normal $\mathbf{n}$ on the boundary $\partial\Omega$; ϵ a small positive parameter; f a nonlinear function of bistable type, a typical example being $f(u) = u - u^3$; and the symbol $\langle \cdot \rangle$ stands for the spatial average over Ω, i.e.,

$$\langle \varphi \rangle := \frac{1}{|\Omega|} \int_\Omega \varphi \, dx.$$

Rubinstein and Sternberg [1)] derived the nonlocal equation (RD) as a *shadow system* for the *viscous Cahn-Hilliard equation* [2),3)]

$$\text{(vCH)} \qquad \begin{cases} \tau u_t = -\triangle[\epsilon^2 \triangle u + f(u) - u_t], \\ \partial u / \partial \mathbf{n} = \partial \triangle u / \partial \mathbf{n} = 0, \end{cases}$$

with respect to the limit operation of the parameter $\tau \to 0$. The function $u = u(t, x)$ represents, e.g., an order parameter or a concentration of one of the components in the mixture at time $t > 0$ and position $x \in \Omega$, and the term $\triangle u_t$ is regarded as a viscous effect. In particular, if the viscous effect is negligible, (vCH) is reduced to the *Cahn-Hilliard equation*

$$\text{(CH)} \qquad \begin{cases} \tau u_t = -\triangle[\epsilon^2 \triangle u + f(u)], \\ \partial u / \partial \mathbf{n} = \partial \triangle u / \partial \mathbf{n} = 0. \end{cases}$$

For (RD) with sufficiently small $\epsilon > 0$, it is known that the dynamics of solution consists of several stages, and is roughly summarized as follows.

(S1) Generation of layers.

The solution with an appropriate initial condition generates sharp *internal transition layer* in a narrow region of $O(\epsilon)$ near a hypersurface, called an *interface*. Such a solution is referred to as a *layer solution*.

(S2) Motion of interfaces (i).

The layer solution begins to move in such a way that the corresponding interface is driven according to a certain motion law.

(S3) Motion of interfaces (ii).

The layer solution then comes to evolve such that the motion of the corresponding interface is governed by another motion law, called the *averaged mean curvature flow*. The interface is driven in such a way that the volume of domain enclosed by itself is preserved and its surface area decreases. After a coarsening process, the interface evolves into a single sphere.

(S4) Motion of bubbles (i).

The layer solution with spherical shape is referred to as the *bubble solution*. The bubble solution drifts with exponentially slow speed, without changing shape, towards the closest point on $\partial\Omega$ from the center of the corresponding sphere.

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(S5) Motion of bubbles (ii).

Once the bubble solution attaches to the boundary $\partial\Omega$, it intersects perpendicularly to $\partial\Omega$ with hemisphere-like shape, and evolves along $\partial\Omega$ by its geometric information.

The dynamics in **(S1)** through **(S3)** was discussed by means of formal asymptotic analysis [1]. For **(S4)**, the existence of bubble motions was rigorously established by Alikakos et al. [4]. Ward [5),6] gave an explicit asymptotic ordinary differential equation for the distance between the center of the bubble and the closest point on $\partial\Omega$ from it. Alikakos et al. [7] derived such an ordinary differential equation for the Cahn-Hilliard equation (CH), and compared the bubble motions for (CH) with those for the nonlocal equation (RD). The dynamics in **(S5)** was studied by Alikakos et al. [8] and Ward [6]. This paper is concerned with the dynamics occurring in the stages **(S2)** and **(S3)**.

2. Internal transition layers and interfaces

In the stages **(S2)** and **(S3)**, the dynamics of layer solutions is approximately captured by a motion law of interface. Such a motion law is called an *interface equation*. Throughout the remaining part of this paper, an interface means

> a smooth, closed hypersurface embedded in $\Omega \subset \mathbb{R}^N$, staying uniformly away from $\partial\Omega$.

The interface Γ separates the whole domain Ω into two subdomains. We denote by Ω^+ one containing $\partial\Omega$ as a part of boundary, and by Ω^- the other:

$$\Omega = \Omega^- \cup \Gamma \cup \Omega^+, \qquad \partial\Omega^- = \Gamma, \qquad \partial\Omega^+ = \partial\Omega \cup \Gamma,$$

and by $\nu(x;\Gamma)$ the unit normal vector on Γ at $x \in \Gamma$ pointing toward the interior of the subdomain Ω^+. We also let the nonlinear function $f(u)$ satisfy the conditions listed below, in which the nonlinearity is regarded as $f(u) - v$, rather than $f(u)$ itself, by introducing an auxiliary variable v for the nonlocal term.

(A1) The function f is smooth on $\mathbb{R}$ and the nullcline $\{(u,v) \,|\, f(u) - v = 0\}$ has exactly three branches of solutions

$$\{(u,v) \,|\, u = h^-(v),\ v \in (\underline{v}, \infty)\},$$
$$\{(u,v) \,|\, u = h^+(v),\ v \in (-\infty, \overline{v})\},$$
$$\{(u,v) \,|\, u = h^0(v),\ v \in (\underline{v}, \overline{v})\},$$

satisfying the following inequalities for each $v \in (\underline{v}, \overline{v})$:

$$h^-(v) < h^0(v) < h^+(v),$$
$$f'(h^\pm(v)) < 0, \quad \text{or equivalently,} \quad h^\pm_v(v) < 0.$$

(A2) For each $v \in (\underline{v}, \overline{v})$, define $\mathbf{J}(v)$ by

$$\mathbf{J}(v) := \int_{h^-(v)}^{h^+(v)} (f(u) - v)\, du.$$

Then there exists a unique point $v^* \in (\underline{v}, \overline{v})$ such that $\mathbf{J}(v^*) = 0$ and $\mathbf{J}'(v^*) < 0$.

2.1 Interface dynamics in (S2).

The dynamics of layer solutions to (RD) in **(S2)** is slow, and is of order $O(\epsilon)$. In order to capture its dynamics in time scale of $O(1)$, we rescale the time t in (RD) as $t \to t/\epsilon$:

$$(\mathrm{RD}') \qquad \begin{cases} \epsilon u^\epsilon_t = \epsilon^2 \triangle u^\epsilon + f(u^\epsilon) - \langle f(u^\epsilon) \rangle, \\ \partial u^\epsilon / \partial \mathbf{n} = 0. \end{cases}$$

Under the assumption **(A1)**, it is known [9] that the following problem, called the *nonlinear eigenvalue problem*, has a unique smooth solution pair $(Q(z;v), c(v))$ for each $v \in (\underline{v}, \overline{v})$ acting as a parameter:

$$(\mathrm{NEP}) \qquad \begin{cases} Q_{zz} + c\,Q_z + f(Q) - v = 0, \\ \lim\limits_{z \to \pm\infty} Q = h^\pm(v), \quad Q|_{z=0} = h^0(v). \end{cases}$$

The functions Q and c are referred to as the *profile* and the *speed* of the traveling wave, respectively. By employing the wave speed $c(v)$, the interface equation in **(S2)** is expressed as

$$(\mathrm{IE}') \qquad \begin{cases} \mathbf{v}(x;\Gamma(t)) = c(v(t)), \\ \dot v(t) = h(v(t);\Gamma(t)) c(v(t)) |\Gamma(t)|, \\ \Gamma(0) = \Gamma_0, \quad v(0) = v_0 \in (\underline{v}, \overline{v}) \end{cases}$$

with

$$h(v;\Gamma) := \frac{h^+(v) - h^-(v)}{h^-_v(v)|\Omega^+| + h^+_v(v)|\Omega^+|}.$$

Here, the scale of time is that of (RD′), the symbol $\mathbf{v}(x;\Gamma(t))$ stands for the normal velocity of $\Gamma(t)$ at $x \in \Gamma(t)$ in ν-direction; $|\Omega^\pm|$ and $|\Gamma|$ are the volume of $\Omega^\pm$ and the surface area of Γ, respectively.

The motion law of interface in **(S2)** was earlier given in [1]. The form, however, was implicit and unsuitable for the circumstantial examination. The explicit form by the interface equation (IE′) was later derived by Okada [10], in which the unique existence of smooth solutions and the stability of the equilibria to (IE′) were successfully established.

For $0 < \epsilon \ll 1$, the dynamics of the solution u^ϵ and the nonlocal effect $\langle f(u^\epsilon)\rangle$ to (RD′) are approximated by that of the solution pair $(\Gamma(t), v(t))$ to (IE′) in the sense that

$$\langle f(u^\epsilon(t,\cdot))\rangle \approx v(t),$$

$$u^\epsilon(t,x) \approx h^\pm(v(t)), \quad x \in \Omega^\pm(t).$$

Note, in particular, that the second property shows the sharp layer structure of u^ϵ near the interface $\Gamma(t)$. Such a characterization in the stage **(S2)** was justified by Okada [10] as follows.

Theorem 1. *Assume that* **(A1)** *and* **(A2)** *are satisfied, and let* (Γ, v) *be the smooth solution pair to* (IE′) *on a time interval* $[0,T]$. *Then there exist* $\epsilon^* > 0$ *and an* ϵ*-family of smooth solutions* u^ϵ *to* (RD′), *defined for* $\epsilon \in (0,\epsilon^*]$, *satisfying*

$$\lim_{\epsilon\to 0}\langle f(u^\epsilon)\rangle = v \text{ uniformly on } [0,T],$$

$$\lim_{\epsilon\to 0} u^\epsilon = h^\pm(v) \text{ uniformly on } \overline{\Omega^\pm_T}\setminus\Gamma^\delta_T \text{ for each } \delta > 0.$$

Here,

$$\Omega^\pm_T := \bigcup_{t\in[0,T]}\{t\}\times\Omega^\pm(t), \quad \Gamma^\delta_T := \bigcup_{t\in[0,T]}\{t\}\times\Gamma(t)^\delta$$

with $\Gamma(t)^\delta := \{x \in \Omega \,|\, \mathrm{dist}\,(x,\Gamma(t)) < \delta\}$, the δ-neighborhood of the interface $\Gamma(t)$.

2.2 Interface dynamics in (S3).

In the stage **(S3)**, the dynamics of layer solutions to (RD) is much slower, compared with that in **(S2)**, which is of order $O(\epsilon^2)$. To capture this in time scale of $O(1)$, it is adequate to rescale the time t in (RD) as $t \to t/\epsilon^2$, and to employ the rescaled version

$$(1) \qquad \begin{cases} \epsilon^2 u^\epsilon_t = \epsilon^2\triangle u^\epsilon + f(u^\epsilon) - \langle f(u^\epsilon)\rangle, \\ \partial u^\epsilon/\partial\mathbf{n} = 0. \end{cases}$$

The corresponding interface equation is known to be the *averaged mean curvature flow*:

$$(2) \qquad \begin{cases} \mathbf{v}(x;\Gamma(t)) = -\kappa(x;\Gamma(t)) + \overline{\kappa}(t), \\ \Gamma(0) = \Gamma_0. \end{cases}$$

Here, the scale of time is that of (1), the symbol $\kappa(x;\Gamma)$ stands for the sum of principal curvatures (the *mean curvature*, for short) of Γ at $x \in \Gamma$, and $\overline{\kappa}$ denotes the average of κ on Γ, i.e.,

$$\overline{\kappa}(t) := \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)}\kappa(x;\Gamma(t))\,dS^{\Gamma(t)}_x,$$

dS^Γ_x being the surface element of Γ at $x \in \Gamma$. We notice that the sign of κ is chosen so that it is positive if the center of the curvature sphere lies in Ω^-. The existence and uniqueness of smooth solutions to the averaged mean curvature flow (2) are well established [11)–14)].

In the previous stage **(S2)**, the interface dynamics by (IE′) approximates the dynamics of layer solution to (RD′) for small $\epsilon > 0$. Then, in this stage **(S3)**,

> does the averaged mean curvature flow (2) approximate the dynamics of layer solution to the nonlocal reaction-diffusion equation (1) for small $\epsilon > 0$?

By a variational method, it was earlier proved in Bronsard and Stoth [15] that the answer to this question is affirmative *for radially symmetric solutions in a spherically symmetric domain*. Later, it was proved by Okada [16] that the answer remains affirmative *without* such restrictions of symmetricity.

Theorem 2. *Assume that* **(A1)** *and* **(A2)** *are satisfied, and let* Γ *be the smooth solution to* (2) *on a time interval* $[0,T]$. *Then there exist* $\epsilon^* > 0$ *and an* ϵ*-family of smooth solutions* u^ϵ *to* (1), *defined for* $\epsilon \in (0,\epsilon^*]$, *satisfying*

$$\lim_{\epsilon\to 0}\langle f(u^\epsilon)\rangle = v^* \text{ uniformly on } [0,T],$$

$$\lim_{\epsilon\to 0} u^\epsilon = h^\pm(v^*) \text{ uniformly on } \overline{\Omega^\pm_T}\setminus\Gamma^\delta_T \text{ for each } \delta > 0.$$

It is in general not so easy to establish this sort of convergence result for nonlocal problems. One reason for the difficulty is that the *method of sub- and super solutions* based on the maximum principle, or comparison principle, is not applicable. Situation is the same even for the most fundamental scalar equation (1), and so it is for the higher order equations such as (vCH) and (CH). To establish the convergence results as in Theorem 1 and Theorem 2, we followed an alternative method, an *approximation method*. This method is based on the singular perturbation method and has been developed as a way to treat boundary/internal layers appearing in local elliptic problems [17)–27)], and in local parabolic problems [28)–31)]. It is expected that the approximation method is also applicable not only for scalar but also for system of (or higher order) nonlocal equations, and a further investigation in this direction will be our future task.

We exhibit in the remaining part of this paper that the approximation method does work for the scalar nonlocal reaction-diffusion equation (1), via the proof of Theorem 2. The procedure of approximation method consists of two parts. The first part is to construct approximate solutions with high degree of accuracy.

Proposition 1 (construction of approximate solutions**).** *Assume that* **(A1)** *and* **(A2)** *are satisfied, and let* Γ *be the smooth solution of* (2) *on a time interval* $[0,T]$*. For each integer* $k \geq 2$*, there exist* $\epsilon^* > 0$ *and an* ϵ*-family of smooth approximate solutions* u_A^ϵ *to* (1)*, defined for* $\epsilon \in (0,\epsilon^*]$*, satisfying*

$$\left\| \epsilon^2 \frac{\partial u_A^\epsilon}{\partial t} - \epsilon^2 \triangle u_A^\epsilon - f(u_A^\epsilon) + \langle f(u_A^\epsilon)\rangle \right\|_{L^\infty} = O(\epsilon^{k+1}),$$

$$\frac{\partial u_A^\epsilon}{\partial \mathbf{n}} = 0,$$

$$\lim_{\epsilon\to 0} \langle f(u_A^\epsilon)\rangle = v^* \ \textit{uniformly on}\ [0,T],$$

$$\lim_{\epsilon\to 0} u_A^\epsilon = h^\pm(v^*) \ \textit{uniformly on}\ \overline{\Omega}_T^\pm \setminus \Gamma_T^\delta \ \textit{for each}\ \delta > 0.$$

Second part is to derive true solutions as a perturbation from the approximate solutions constructed in Proposition 1.

Proposition 2 (true solutions near approximate ones**).** *For each integer* $k \geq 2$*, let* u_A^ϵ *be the* ϵ*-family of smooth approximate solutions to* (1)*, defined for* $\epsilon \in (0,\epsilon^*]$*, satisfying the properties stated in Proposition* 1*. Then there exists an* ϵ*-family of smooth true solutions* u^ϵ *to* (1)*, defined for* $\epsilon \in (0,\epsilon^*]$*, such that*

$$\|u^\epsilon - u_A^\epsilon\|_{L^\infty} = O(\epsilon^{k-1}).$$

Theorem 2 immediately follows from two propositions above.

3. Outline of proof of Proposition 1

Let us now recast the equation in (1), by introducing an auxiliary variable v, as

(3a) $$\epsilon^2 u_t^\epsilon = \epsilon^2 \triangle u^\epsilon + f(u^\epsilon) - v^\epsilon,$$

(3b) $$v^\epsilon - \langle f(u^\epsilon)\rangle = 0,$$

and treat the equation (3a) as a scalar equation with v^ϵ being regarded as a parameter.

Let $\mathcal{M}$ be a fixed $(N-1)$-dimensional reference manifold, and $\gamma(t,\cdot) : \mathcal{M} \to \Omega$ a representation of $\Gamma(t)$. We define the ϵ-dependent interface $\Gamma^\epsilon(t)$ as a level set of the solution u^ϵ. The transition layer is expected to develop near $\{x \in \Omega \,|\, u^\epsilon(t,x) \approx h^0(v^*)\}$, and without loss of generality, we may assume $h^0(v^*) := 0$ by an appropriate translation. From this, we set

$$\Gamma^\epsilon(t) := \{x \in \Omega \,|\, u^\epsilon(t,x) = 0\}.$$

We also expect that $\Gamma^\epsilon(t)$ is expressed as the graph of a smooth function over $\Gamma(t)$:

$$\Gamma^\epsilon(t) = \{x \in \Omega \,|\, x = \gamma(t,y) + \epsilon R^\epsilon(t,y)\nu(t,y),\ y \in \mathcal{M}\},$$

where R^ϵ is a priori unknown and to be determined.

Let us decompose the domain Ω by the interface $\Gamma^\epsilon(t)$ as

$$\Omega = \Omega^{\epsilon,-}(t) \cup \Gamma^\epsilon(t) \cup \Omega^{\epsilon,+}(t)$$

and consider the following approximation problem associated with (3a)

$$\epsilon^2 u_t^{\epsilon,\pm} = \epsilon^2 \triangle u^{\epsilon,\pm} + f(u^{\epsilon,\pm}) - v^\epsilon$$

for $t > 0$, $x \in \Omega^{\epsilon,\pm}(t)$, with the boundary conditions

$$u^{\epsilon,\pm}|_{\Gamma^\epsilon(t)} = 0, \quad \frac{\partial u^{\epsilon,+}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega.$$

Substituting formal expansions

$$R^\epsilon = R^1 + \epsilon R^2 + \epsilon^2 R^3 + \cdots, \quad v^\epsilon = v^0 + \epsilon v^1 + \epsilon^2 v^2 + \cdots$$

into the equations and the boundary conditions above, we obtain the formal *outer approximate solutions* $u_{A,\mathrm{OUT}}^{\epsilon,\pm}$ (giving good approximation on the outer regions $\Omega^\pm \setminus \Gamma(t)^\delta$) and *inner approximate solutions* $u_{A,\mathrm{IN}}^{\epsilon,\pm}$ (giving good approximation on the inner region $\Gamma(t)^\delta$). We note, at this stage, that our outer and inner approximate solutions still depend on the unknown data $(\Gamma, R^\epsilon; v^\epsilon)$.

Once the formal approximate solutions are obtained, we impose on them C^1*-matching conditions*

$$\frac{\partial u_{A,\mathrm{IN}}^{\epsilon,-}}{\partial \nu} = \frac{\partial u_{A,\mathrm{IN}}^{\epsilon,+}}{\partial \nu} \quad \text{on } \Gamma^\epsilon(t),$$

which guarantee that the inner solution is smooth across the level-set interface $\Gamma^\epsilon(t)$. This condition gives rise to a series of equations for $(\Gamma, R^\epsilon; v^\epsilon)$; the lowest order (0-th order) equation is $v^0 = v^*$, which means that in this stage **(S3)** the interface dynamics in the previous stage **(S2)** is in equilibrium. The first order equation is

$$\gamma_t \cdot \nu = -\kappa + c'(v^*)v^1,$$

and j-th ($j \geq 2$) order equation is

$$R_t^{j-1} = \left(\triangle_{\mathcal{M}}^\Gamma + \sum_{i=1}^{N-1} \kappa_i^2\right) R^{j-1} + c'(v^*)v^j$$
$$+ \text{ lower order terms depend only on } (\Gamma, R^1, \ldots, R^{j-2}; v^*, v^1, \ldots, v^{j-1}),$$

where $\triangle^{\Gamma}_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$ induced from those on $\Gamma(t)$ by $\gamma(t,\cdot)$. In order to determine $(\Gamma, R^1, \ldots; v^1, v^2, \ldots)$, we need another series of equations coupled with the equations above.

Let us now move on to the second equation (3b). We can easily find that $v^\epsilon - \langle f(u^\epsilon)\rangle$ in (3b) is recast as

$$v^\epsilon - \langle f(u^\epsilon)\rangle =$$

(4a) $$\frac{1}{|\Omega|}\int_{\Gamma(t)^\delta}(\epsilon^2\triangle u^\epsilon - \epsilon^2 u^\epsilon_t)\,dx$$

(4b) $$+\frac{1}{|\Omega|}\int_{\Omega\setminus\Gamma(t)^\delta}(\epsilon^2\triangle u^\epsilon - \epsilon^2 u^\epsilon_t)\,dx.$$

We recall that our inner and outer approximate solutions $u^{\epsilon,\pm}_{A,\mathrm{IN}}$ and $u^{\epsilon,\pm}_{A,\mathrm{OUT}}$ still depend on the unknown data $(\Gamma, R^\epsilon; v^\epsilon)$. Substituting $u^{\epsilon,\pm}_{A,\mathrm{IN}}$ and $u^{\epsilon,\pm}_{A,\mathrm{OUT}}$ into (4a) and (4b), respectively, and constraining $v^\epsilon - \langle f(u^\epsilon)\rangle := 0$ so that (3b) is satisfied, we obtain another series of equations for $(\Gamma, R^\epsilon; v^\epsilon)$; the first order equation is

$$c'(v^*)v^1 = \frac{1}{|\Gamma|}\int_{\mathcal{M}}\kappa\,dS^\Gamma_y,$$

and the j-th ($j \geq 2$) order equation is

$$c'(v^*)v^j = -\frac{1}{|\Gamma|}\int_{\mathcal{M}}\left(\sum_{i=1}^{N-1}\kappa_i^2 + \kappa(\gamma_t\cdot\nu)\right)R^{j-1}\,dS^\Gamma_y$$
$$+ \text{ lower order terms depend only on } (\Gamma, R^1, \ldots, R^{j-2}; v^*, v^1, \ldots, v^{j-1}).$$

We now find that C^1-matching condition and the nonlocal constraint give rise to the following series of parabolic problems:

(5a)
$$\gamma_t\cdot\nu = -\kappa + \frac{1}{|\Gamma|}\int_{\mathcal{M}}\kappa\,dS^\Gamma_y,$$

(5b)
$$R^{j-1}_t = \left(\triangle^\Gamma_{\mathcal{M}} + \sum_{i=1}^{N-1}\kappa_i^2\right)R^{j-1}$$
$$-\frac{1}{|\Gamma|}\int_{\mathcal{M}}\left(\sum_{i=1}^{N-1}\kappa_i^2 + \kappa(\gamma_t\cdot\nu)\right)R^{j-1}\,dS^\Gamma_y$$
$$+ \text{ lower order terms depend only on } (\Gamma, R^1, \ldots, R^{j-2}; v^*, v^1, \ldots, v^{j-1}).$$

The first equation (5a) is nothing but the averaged mean curvature flow. Thus, once a smooth initial interface is given, it is guaranteed that there exists unique smooth solution Γ to (5a) on a finite time interval [11)–14)]. The equation (5b) is a nonlocal nonhomogeneous linear parabolic equation of the following form

$$R_t = \left(\triangle^\Gamma_{\mathcal{M}} + \sum_{i=1}^{N-1}\kappa_i^2\right)R + \int_{\mathcal{M}} a^\Gamma R\,dy + b,$$

where a^Γ is a function depending only on Γ and b a nonhomogereous term. This is expressed as

$$R_t(t,y) = \mathbf{A}(t,y)R(t,y) + b(t,y),$$

and the generator of $\mathbf{A}$ is sectorial because the linear differential operator $\triangle^\Gamma_{\mathcal{M}} + \sum_{i=1}^{N-1}\kappa_i^2$, called the *Jacobi operator*, generates a sectorial operator while the linear nonlocal effect $\int_{\mathcal{M}} a^\Gamma R\,dy$ defines a bounded operator. Therefore, by the abstract theory for evolution equations [32)], it is ensured that there exists a unique smooth solution R on a finite time interval, provided that Γ, b and initial data are all smooth. Thus, the series of equations for R^j are successively solvable and we obtain the desired approximate solution as in Proposition 1.

4. Outline of proof of Proposition 2

In what follows, where no danger of confusion will arise, we employ the same symbol M to denote positive constants independent of $\epsilon > 0$ which could differ from line to line.

For each $t \in [0, T]$, let $\mathcal{L}^\epsilon(t)$ be the linearized operator of (1) around the approximate solution u^ϵ_A:

$$\mathcal{L}^\epsilon(t)\varphi := \triangle\varphi + \frac{1}{\epsilon^2}\Big[f'(u^\epsilon_A(t,\cdot))\varphi - \langle f'(u^\epsilon_A(t,\cdot))\varphi\rangle\Big].$$

By introducing a scaling parameter $s \in \mathbb{R}$, which is to be determined, we rescale the time t in $\mathcal{L}^\epsilon(t)$ by

(6) $$t = \epsilon^s\tau,$$

and seek a true solution u^ϵ of (1) with the following form:

(7) $$u^\epsilon(\epsilon^s\tau,\cdot) = u^\epsilon_A(\epsilon^s\tau,\cdot) + \varphi^\epsilon(\tau)(\cdot), \quad \tau \in [0, T/\epsilon^s].$$

Our equation in (1) is recast as an evolution equation

(8) $$\dot\varphi^\epsilon(\tau) = \mathcal{A}^\epsilon(\tau)\varphi^\epsilon(\tau) + \mathcal{N}^\epsilon(\tau, \varphi^\epsilon(\tau)) + \mathcal{R}^\epsilon(\tau),$$

where "dot" stands for the derivative with respect to the variable τ; $\mathcal{A}^\epsilon(\tau)\varphi := \epsilon^s\mathcal{L}^\epsilon(\epsilon\tau)\varphi$, $\mathcal{N}^\epsilon(\tau,\varphi)$ and $\mathcal{R}^\epsilon(\tau)$ are the linear, the nonlinear and the remainder parts, respectively, satisfying

(9a) $$\epsilon^{2-s}\mathcal{N}^\epsilon(\tau,\varphi) = O(|\varphi|^2),$$

(9b) $$\|\mathcal{R}^\epsilon(\tau)\|_{L^\infty} = O(\epsilon^{s+k-1}),$$

for $\tau \in [0, T/\epsilon^s]$.

We now decompose $L^2(\Omega)$ as $L^2(\Omega) = \mathbf{M}\oplus\mathbf{M}^\perp$, where $\mathbf{M}$ and $\mathbf{M}^\perp$ stand for the space consisting of zero-average functions and the orthogonal complement of $\mathbf{M}$

spanned by the constant function $|\Omega|^{-1/2}$, respectively. According to this decomposition, we also decompose the function $\varphi^\epsilon(\tau)$ in (8) as

(10) $$\varphi^\epsilon(\tau)(\cdot) = \varphi_1^\epsilon(\tau)(\cdot) + \varphi_2^\epsilon(\tau),$$

where $\varphi_1^\epsilon(\tau) \in \mathbf{M}$ and $\varphi_2^\epsilon(\tau) \in \mathbf{M}^\perp$. Then, the equation (8) is equivalent to the following system:

(11a)
$$\dot{\varphi}_1^\epsilon(\tau) = \mathcal{A}^\epsilon(\tau)\varphi_1^\epsilon(\tau) + \mathcal{N}^\epsilon\big(\tau, \varphi_1^\epsilon(\tau) + \varphi_2^\epsilon(\tau)\big) + \hat{\mathcal{R}}^\epsilon\big(\tau, \varphi_2^\epsilon(\tau)\big),$$

(11b)
$$\dot{\varphi}_2^\epsilon(\tau) = \langle \mathcal{R}^\epsilon(\tau)\rangle.$$

Here, $\hat{\mathcal{R}}^\epsilon(\tau, \varphi_2)$ ($\varphi_2 \in \mathbf{M}^\perp$) is defined by

(12)
$$\hat{\mathcal{R}}^\epsilon(\tau,\varphi_2) := \mathcal{R}^\epsilon(\tau) - \langle \mathcal{R}^\epsilon(\tau)\rangle + \epsilon^{s-2}\Big[f'\big(u_A^\epsilon(\epsilon^s\tau,\cdot)\big) - \big\langle f'\big(u_A^\epsilon(\epsilon^s\tau,\cdot)\big)\big\rangle\Big]\varphi_2.$$

Note that (11a) is the *evolution equation* and (11b) the *ordinary differential equation*.

In order to deal with the evolution equation (11a), let us now set up some appropriate function spaces. Let $p \geq 2$ and we define the basic space by

(13) $$X_0^\epsilon := L^p(\Omega) \cap \mathbf{M}$$

and the domain of $\mathcal{A}^\epsilon(\tau)$ by

$$X_1^\epsilon := W^{2,p}_{\epsilon,\mathcal{B}}(\Omega) \cap \mathbf{M},$$

where $W^{2,p}_{\epsilon,\mathcal{B}}(\Omega)$ is the same as the usual Sobolev space

$$W^{2,p}_{\mathcal{B}}(\Omega) := \{u \in W^{2,p}(\Omega) \,|\, \partial u/\partial \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

as a set, with the weighted norm

(14) $$\|u\|_{W^{2,p}_{\epsilon,\mathcal{B}}} := \|u\|_{L^p} + \epsilon^{\frac{s}{2}}\|Du\|_{L^p} + \epsilon^s\|D^2u\|_{L^p}.$$

In the sequel, some weighted norms and embedding properties are employed. We notice here that the weighted norms are introduced to make the embedding constants *independent* of $\epsilon > 0$.

For $\alpha \in (0,1)$, let X_α^ϵ be the real interpolation spaces between X_0^ϵ and X_1^ϵ

(15) $$X_\alpha^\epsilon := (X_0^\epsilon, X_1^\epsilon)_{\alpha,p}$$

endowed with the norms $\|\cdot\|_\alpha$, where $(\cdot,\cdot)_{\alpha,p}$ stands for the standard real interpolation method (functor). Note that X_α^ϵ enjoy the continuous embedding properties

$$0 \leq \alpha < \beta \leq 1 \implies \begin{array}{l} X_\beta^\epsilon \hookrightarrow X_\alpha^\epsilon, \\ \|u\|_\alpha \leq M\|u\|_\beta \quad (u \in X_\beta^\epsilon). \end{array}$$

We also set up weighted Hölder spaces $C^\alpha_{\epsilon,p}(\overline{\Omega})$ for $\alpha \in (0,1)$, which are the same as the usual Hölder spaces $C^\alpha(\overline{\Omega})$ as sets, with the weighted norm

(16) $$\|u\|_{C^\alpha_{\epsilon,p}} := \epsilon^{\frac{sN}{2p}}\|u\|_{L^\infty} + \epsilon^{\frac{s}{2}(\alpha+\frac{N}{p})}[u]_{C^\alpha},$$

where $[u]_{C^\alpha}$ is the Hölder seminorm defined by

$$[u]_{C^\alpha} := \sup_{\substack{x,x'\in\overline{\Omega} \\ x\neq x'}} \frac{|u(x)-u(x')|}{|x-x'|^\alpha}.$$

Notice that, if the relation $2\alpha - N/p > \beta$ is valid for some $\alpha,\, \beta \in (0,1)$, then X_α^ϵ is continuously embedded in $C^\beta_{\epsilon,p}(\overline{\Omega})$:

(17)
$$2\alpha - \frac{N}{p} > \beta \implies \begin{array}{l} X_\alpha^\epsilon \hookrightarrow C^\beta_{\epsilon,p}(\overline{\Omega}), \\ \|u\|_{C^\beta_{\epsilon,p}} \leq M\|u\|_\alpha \quad (u \in X_\alpha^\epsilon). \end{array}$$

We simply denote by $\|B\|_{\alpha,\beta}$ the operator norm of a bounded linear operator $B : X_\alpha^\epsilon \to X_\beta^\epsilon$.

It is easy to verify that the linear operator $\mathcal{L}^\epsilon(t)$ under the Neumann boundary condition is formally self-adjoint in $L^2(\Omega)\cap\mathbf{M}$, and therefore eigenvalues are real. We also obtain by the variational characterization for the principal eigenvalue λ^ϵ of $\mathcal{L}^\epsilon(t)$ that

$$\begin{aligned}\lambda^\epsilon &= \sup_{\substack{\varphi\in H^1(\Omega)\\ \varphi\neq 0,\, \langle\varphi\rangle = 0}} \frac{\int_\Omega -|\nabla\varphi|^2 + \epsilon^{-2}f'(u_A^\epsilon)|\varphi|^2\,dx}{\|\varphi\|^2_{L^2(\Omega)}} \\ &\leq \sup_{\substack{\varphi\in H^1(\Omega)\\ \varphi\neq 0}} \frac{\int_\Omega -|\nabla\varphi|^2 + \epsilon^{-2}f'(u_A^\epsilon)|\varphi|^2\,dx}{\|\varphi\|^2_{L^2(\Omega)}}.\end{aligned}$$

This says that λ^ϵ is estimated from above by the principal eigenvalue of the *linearized Allen-Cahn operator* $\triangle + \epsilon^{-2}f'(u_A^\epsilon)$. On the other hand, according to the results established by Alikakos et al. [33] and Chen [34], the principal eigenvalue of the linearized Allen-Cahn operator is bounded above for $\epsilon > 0$ and $t \in [0,T]$. Thus we have $\lambda^\epsilon \leq \lambda_*$ for some $\lambda_* > 0$.

By considering the resolvent equation in $L^2(\Omega)\cap\mathbf{M}$, it turns out that $\mathcal{L}^\epsilon(t)$ is a sectorial operator and that a resolvent estimate is valid in $L^2(\Omega)$. Along the line of arguments in Tanabe [35], the resolvent estimate can be modified from L^2-version to L^p-version ($p > 2$). Rescaling the time t in $\mathcal{L}^\epsilon(t)$ as in (6), we find that $\mathcal{A}^\epsilon(\tau)$ is a sectorial operator for each $\tau \in [0, T/\epsilon^s]$ satisfying the resolvent estimate

(18)
$$\|(\lambda - \mathcal{A}^\epsilon(\tau))^{-1}\|_{0,0} \leq \frac{M_*}{|\lambda - \epsilon^s\lambda_*|}, \quad \lambda \in S_* \subset \rho(\mathcal{A}^\epsilon(\tau))$$

for some $\theta_* \in (0, \pi/2)$ and $M_* > 0$. Here,

$$S_* := \{\lambda \in \mathbb{C} \mid \lambda \neq \epsilon^s \lambda_*,\ |\arg(\lambda - \epsilon^s \lambda_*)| < \pi/2 + \theta_*\}.$$

On the other hand, we note that the operator $\mathcal{A}^\epsilon(\tau) - \mathcal{A}^\epsilon(\sigma)$ consists of a multiplication operator and an integral operator. In particular, it does not involve any differential operator. Thanks to this fact, we can easily show that there exists a constant $M > 0$ such that for $\alpha \in [0, 1/2)$

$$\|\mathcal{A}^\epsilon(\tau) - \mathcal{A}^\epsilon(\sigma)\|_{1,\alpha} \leq M\epsilon^s(\tau - \sigma)$$

holds for sufficiently small $\epsilon > 0$, $s \geq 4$ and $0 \leq \sigma \leq \tau \leq T/\epsilon^s$. Moreover, from the resolvent estimate (18), we find that for $0 \leq \alpha \leq \beta \leq 1$, there exists a constant $M = M(\alpha, \beta) > 0$ such that the following estimate is valid:

$$\|e^{(\tau-\sigma)\mathcal{A}^\epsilon(\sigma)}\|_{\alpha,\beta} \leq M(\tau - \sigma)^{\alpha-\beta} e^{\epsilon^s \lambda_*(\tau-\sigma)}.$$

Therefore, we can prove that for $0 \leq \alpha \leq \beta \leq 1$ with $(\alpha, \beta) \neq (0, 1)$, there exists a constant $K > 0$ such that the *evolution operator* $\Phi^\epsilon(\tau, \sigma) : X_\alpha^\epsilon \to X_\beta^\epsilon$ $(0 \leq \sigma \leq \tau \leq T/\epsilon^s)$ associated with the family $\{\mathcal{A}^\epsilon(\tau)\}$ satisfies the estimate

$$(19) \quad \|\Phi^\epsilon(\tau, \sigma)\|_{\alpha,\beta} \leq M(\tau - \sigma)^{\alpha-\beta} e^{\epsilon^s(\lambda_* + K)(\tau-\sigma)}$$

for $s \geq 4$ and small $\epsilon > 0$.

Applying the variation of constants formula to (11), we obtain

(20a)

$$\varphi_1^\epsilon(\tau) = \Phi^\epsilon(\tau, 0)\varphi_1^\epsilon(0) + \int_0^\tau \Phi^\epsilon(\tau, \sigma)\mathcal{N}^\epsilon(\sigma, \varphi_1^\epsilon(\sigma) + \varphi_2^\epsilon(\sigma))\, d\sigma + \int_0^\tau \Phi^\epsilon(\tau, \sigma)\hat{\mathcal{R}}^\epsilon(\sigma, \varphi_2^\epsilon(\sigma))\, d\sigma,$$

(20b)

$$\varphi_2^\epsilon(\tau) = \varphi_2^\epsilon(0) + \int_0^\tau \langle \mathcal{R}^\epsilon(\sigma) \rangle\, d\sigma.$$

The existence and uniqueness of smooth solutions is well established, and therefore our task is only to have an estimate for the solution φ^ϵ to (8) by employing those for φ_1^ϵ and φ_2^ϵ in (20).

We recall that k, s, p and α are parameters related to the accuracy degree of approximations (cf. Proposition 1), the scaling of time (cf. (6)), the basic space (cf. (13)) and the interpolation spaces (cf. (15)), respectively. We now let

(21a) $\quad k \geq 2,$

(21b) $\quad s := 4,$

(21c) $\quad p > 2N,$

(21d) $\quad \alpha \in (3/4, 1),$

and choose $\varphi^\epsilon(0) = \varphi_1^\epsilon(0) + \varphi_2^\epsilon(0)$ so small that

(22a) $\quad \|\varphi_1^\epsilon(0)\|_\alpha = O(\epsilon^{k+1}),$

(22b) $\quad |\varphi_2^\epsilon(0)| = O(\epsilon^{k+1}).$

Let us first treat (20b) together with (22b). The estimate (9b) with (21b) yields that

$$|\varphi_2^\epsilon(\tau)| \leq |\varphi_2^\epsilon(0)| + \int_0^\tau \|\mathcal{R}^\epsilon(\sigma)\|_{L^\infty}\, d\sigma = O(\epsilon^{k+1}) + O(\epsilon^{k+3}) \cdot T/\epsilon^4.$$

Therefore, the solution $\varphi_2^\epsilon(\tau)$ of (11b) with (22b) satisfies

$$(23) \quad |\varphi_2^\epsilon(\tau)| = O(\epsilon^{k-1}), \quad \tau \in [0, T/\epsilon^4].$$

Substituting the solution $\varphi_2^\epsilon(\tau)$ with (23) into (20a), we move on to estimating φ_1^ϵ. Since p and α are chosen so that (21c) and (21d), respectively, it holds that

$$2\alpha - \frac{N}{p} > 2 \cdot \frac{3}{4} - \frac{1}{2} = 1,$$

and the embedding relations in (17) are fulfilled for $\beta \in (0, 1)$ chosen arbitrarily. Hence, by (9a) and (16) with (21b), (23) and $X_\alpha^\epsilon \hookrightarrow X_0^\epsilon$, we have the following estimates for $\sigma \in [0, T/\epsilon^4]$:

$$\begin{aligned}
&\|\mathcal{N}^\epsilon(\sigma, \varphi_1^\epsilon(\sigma) + \varphi_2^\epsilon(\sigma))\|_0 \\
&\leq M\epsilon^2 \|\varphi_1^\epsilon + \varphi_2^\epsilon\|_{L^\infty} \|\varphi_1^\epsilon + \varphi_2^\epsilon\|_0 \\
&\leq M\epsilon^2 \left(\|\varphi_1^\epsilon\|_{L^\infty} + |\varphi_2^\epsilon|\right)\left(\|\varphi_1^\epsilon\|_0 + |\varphi_2^\epsilon|\right) \\
&\leq M\epsilon^2 \left(\epsilon^{-\frac{2N}{p}} \|\varphi_1^\epsilon\|_\alpha + \epsilon^{k-1}\right)\left(\|\varphi_1^\epsilon\|_\alpha + \epsilon^{k-1}\right) \\
&\leq M\left(\epsilon^{2(1-\frac{N}{p})} \|\varphi_1^\epsilon\|_\alpha^2 + \epsilon^{k+1-\frac{2N}{p}} \|\varphi_1^\epsilon\|_\alpha + \epsilon^{2k}\right).
\end{aligned}$$

Moreover, employing (9b), (23) together with (21b) in (12), we have for $\sigma \in [0, T/\epsilon^4]$,

$$\|\hat{\mathcal{R}}^\epsilon(\sigma,\varphi_2^\epsilon)\|_0$$
$$\le \|\mathcal{R}^\epsilon - \langle\mathcal{R}^\epsilon\rangle\|_0$$
$$+ \epsilon^2 \left\|f'\big(u_A^\epsilon(\epsilon^s\sigma)\big) - \big\langle f'\big(u_A^\epsilon(\epsilon^s\sigma)\big)\big\rangle\right\|_0 |\varphi_2^\epsilon|$$
$$\le 2\Big(\|\mathcal{R}^\epsilon\|_{L^\infty} + \epsilon^2 \left\|f'\big(u_A^\epsilon(\epsilon^2\sigma)\big)\right\|_{L^\infty} |\varphi_2^\epsilon|\Big)$$
$$= O(\epsilon^{k+3}) + \epsilon^2\, O(1)\, O(\epsilon^{k-1})$$
$$\le M\epsilon^{k+1}.$$

Using these estimates in (20), we find that

(24)
$$\|\varphi_1^\epsilon(\tau)\|_\alpha$$
$$\le \|\Phi^\epsilon(\tau,0)\|_{\alpha,\alpha}\|\varphi_1^\epsilon(0)\|_\alpha$$
$$+ M\epsilon^{2\left(1-\frac{N}{p}\right)} \int_0^\tau \|\Phi^\epsilon(\tau,\sigma)\|_{0,\alpha}\|\varphi_1^\epsilon(\sigma)\|_\alpha^2\, d\sigma$$
$$+ M\epsilon^{k+1-\frac{2N}{p}} \int_0^\tau \|\Phi^\epsilon(\tau,\sigma)\|_{0,\alpha}\|\varphi_1^\epsilon(\sigma)\|_\alpha\, d\sigma$$
$$+ M\epsilon^{k+1} \int_0^\tau \|\Phi^\epsilon(\tau,\sigma)\|_{0,\alpha}\, d\sigma,$$

where the inequality $2k > k+1$ (under (21a)) has been employed to get the last term.

Let $r^\epsilon(\tau)$ be the function defined by

(25) $$r^\epsilon(\tau) := \|\varphi_1^\epsilon(\tau)\|_\alpha\, e^{-\epsilon^4(\lambda_*+K)\tau}, \quad \tau \in [0, T/\epsilon^4].$$

Then, by the estimates (19) with (21b), we can compute (24) in terms of $r^\epsilon(\tau)$ so that

(26)
$$r^\epsilon(\tau)$$
$$\le M\Big(r^\epsilon(0) + e^{(\lambda*+K)T}\epsilon^{2\left(1-\frac{N}{p}\right)} \times \int_0^\tau (\tau-\sigma)^{-\alpha} r^\epsilon(\sigma)^2\, d\sigma$$
$$+ \epsilon^{k+1-\frac{2N}{p}} \int_0^\tau (\tau-\sigma)^{-\alpha} r^\epsilon(\sigma)\, d\sigma$$
$$+\epsilon^{k+1}\frac{T^{1-\alpha}}{1-\alpha}\,\epsilon^{-4(1-\alpha)}\Big)$$
$$\le M\Big(r^\epsilon(0) + \epsilon^{2\left(1-\frac{N}{p}\right)} \int_0^\tau (\tau-\sigma)^{-\alpha} r^\epsilon(\sigma)^2\, d\sigma$$
$$+ \epsilon^{k+1-\frac{2N}{p}} \int_0^\tau (\tau-\sigma)^{-\alpha} r^\epsilon(\sigma)\, d\sigma + \epsilon^{k+4\alpha-3}\Big).$$

By (22a), we have

(27) $$r^\epsilon(0) = \|\varphi_1^\epsilon(0)\|_\alpha = O(\epsilon^{k+1}).$$

Then from the continuity of $r^\epsilon(\tau)$ it follows that

(28) $$r^\epsilon(\tau) \le \epsilon^k$$

for small $\tau > 0$. Setting

$$T^\epsilon := \sup\{\tau \in [0, T/\epsilon^4]\,|\, r^\epsilon(\sigma) \le \epsilon^k \text{ for all } \sigma \in [0,\tau]\},$$

we have one of the alternatives

$$r^\epsilon(T^\epsilon) = \epsilon^k \quad\text{or}\quad T^\epsilon = T/\epsilon^4.$$

Assuming the former situation is realized, we can compute by employing (27) in (26) so that

(29)
$$\epsilon^k = r^\epsilon(T^\epsilon)$$
$$\le M\Big(\epsilon^{k+1} + \epsilon^{2\left(1-\frac{N}{p}\right)}\epsilon^{2k}\frac{T^{1-\alpha}}{1-\alpha}\,\epsilon^{-4(1-\alpha)}$$
$$+ \epsilon^{k+1-\frac{2N}{p}}\epsilon^k\frac{T^{1-\alpha}}{1-\alpha}\,\epsilon^{-4(1-\alpha)} + \epsilon^{k+4\alpha-3}\Big)$$
$$\le \epsilon^k\Big(M\epsilon + \frac{MT^{1-\alpha}}{1-\alpha}\,\epsilon^{k-2+4\alpha-\frac{2N}{p}}$$
$$+ \frac{MT^{1-\alpha}}{1-\alpha}\,\epsilon^{k-3+4\alpha-\frac{2N}{p}} + M\epsilon^{4\alpha-3}\Big).$$

Noting our way of choice of parameters in (21), we have

$$k-2+4\alpha-\frac{2N}{p} > 2-2+4\cdot\frac{3}{4}-2\cdot\frac{1}{2} = 2 > 0,$$
$$k-3+4\alpha-\frac{2N}{p} > 2-1 = 1 > 0,$$
$$4\alpha-3 > 4\cdot\frac{3}{4}-3 = 0.$$

Thus for sufficiently small $\epsilon > 0$, (29) implies

$$\epsilon^k \le \frac{\epsilon^k}{2},$$

which is a contradiction. Hence, the latter case is realized, namely, (28) is valid for $\tau \in [0, T/\epsilon^4]$, and by (25) we have

$$\|\varphi_1^\epsilon(\tau)\|_\alpha \le Me^{(\lambda_*+K)T}\epsilon^k = O(\epsilon^k), \quad \tau \in [0, T/\epsilon^4].$$

By employing (16) and (17), it follows that

(30) $$\|\varphi_1^\epsilon(\tau)\|_{L^\infty} = O(\epsilon^{k-\frac{2N}{p}}), \quad \tau \in [0, T/\epsilon^4].$$

Combining (23) and (30) in (10), we have

$$\|\varphi^\epsilon(\tau)\|_{L^\infty} \le \|\varphi_1^\epsilon(\tau)\|_{L^\infty} + |\varphi_2^\epsilon(\tau)|$$
$$= O(\epsilon^{k-\frac{2N}{p}}) + O(\epsilon^{k-1})$$
$$= O(\epsilon^{k-1}), \quad \tau \in [0, T/\epsilon^4].$$

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